# Caustic free completion of pressureless perfect fluid and k-essence

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- Gradient or tachyon instabilities.

- Caustic singularities.

E. g. Pressureless perfect fluid, k-essence, ghost condensate.

# Pressureless perfect fluid

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} = 0 \qquad \text{Euler equation}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \qquad \text{Continuity equation}$$

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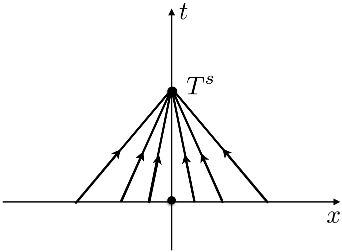
Consider a simple 1D example

$$v(t=0) = -\frac{x}{T^s} \Longrightarrow v = -\frac{x}{T^s - t}$$

NB: Realistically $v = -\frac{x}{T^s - t} + \mathcal{O}\left(\frac{x^2}{[T^s - t]^2}\right)$

still $\boxed{\frac{\partial v}{\partial x} = -\frac{1}{T^s - t}\ (x=0) \to \infty}$

# Caustic formation: characteristics cross



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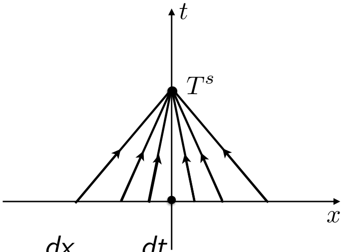


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No caustic singularities provided that $v = \text{const}$

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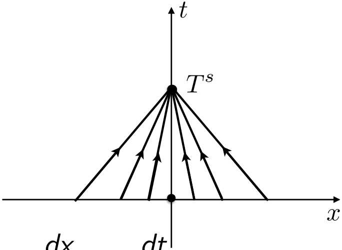


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Natural regularization in particle models.

Pressureless perfect fluid is more generic.

Horava'09 Mukohyama'09 Chamseddine and Mukhanov'13 Lim et al'10

# k-essence

$$S = \int d^4x \sqrt{-g} \mathcal{L}(X, \varphi) \qquad X = \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi$$

Armendariz-Picon et *al*'99 '00

- Originally designed to explain accelerated expansion of the Universe.
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Super-/sub-luminality Garriga and Mukhanov'99,

$$c_s^2 = \frac{1}{1 + 2X\frac{\mathcal{L}_{XX}}{\mathcal{L}_X}}$$

Superluminality is not a threat to causality Babichev et *al*'07.

May cause problems at quantum level Adams et *al*'06.

## k-essence: caustic singularities! Babichev'16

Shift-symmetric Lagrangians in 1+1 space-time $S = \int d^2x \mathcal{L}(X)$

$$\left(\mathcal{L}_X \eta^{\mu\nu} + \mathcal{L}_{XX} \partial^\mu \varphi \partial^\nu \varphi\right) \partial_\mu \partial_\nu \varphi = 0 \, .$$

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Define variables $\boxed{\tau = \dot{\varphi}} \equiv \frac{d\varphi}{dt}$ and $\boxed{\chi = \varphi'} \equiv \frac{d\varphi}{dx}$

$$A\dot{\tau} + 2B\tau' + C\chi' = 0 \qquad A, B, C = F(\tau, \chi) \, .$$

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Consider an arbitrary curve in (x,t) plane defined by parameter $\sigma$.

$$\frac{A}{t_\sigma}\tau_\sigma + \frac{C}{x_\sigma}\chi_\sigma - \frac{\tau'}{\xi}\left(A\xi^2 - 2B\xi + C\right) = 0 \qquad \boxed{\xi \equiv \frac{x_\sigma}{t_\sigma}}.$$

Characteristics: $A\xi^2 - 2B\xi + C = 0$

# Characteristics

$$\frac{dx}{d\sigma_\pm} = \xi_\pm(\tau,\chi)\frac{dt}{d\sigma_\pm} \qquad \frac{d\tau}{d\sigma_\pm} + \xi_\mp \frac{d\chi}{d\sigma_\pm} = 0\ .$$

$$\xi_\pm = \frac{v \pm c_s}{1 \pm v \cdot c_s} \qquad v = -\frac{\chi}{\tau}$$

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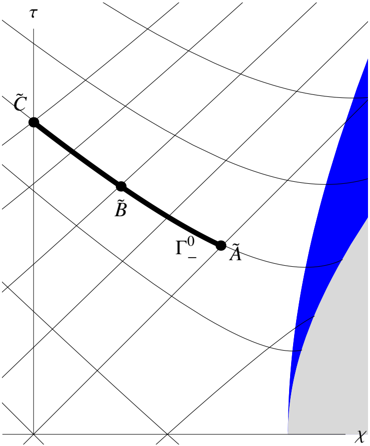
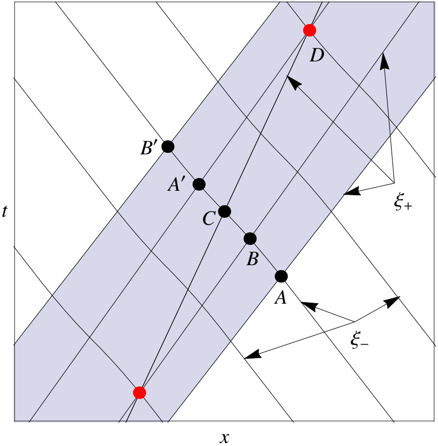
NB Pressureless perfect fluid: $c_s = 0 \Longrightarrow \xi_\pm = v$

NNB No caustic singularities provided that $c_s = 1 \Longrightarrow \xi_\pm = \text{const}$

- $\mathcal{L} = X \Longrightarrow c_s = 1 \Longrightarrow \xi_\pm = \pm 1$
- Traveling wave $\varphi = \varphi(t \pm x)$
  $\tau = \pm\chi \Longrightarrow \xi_\pm = \pm 1$, or $X = \frac{1}{2}(\tau^2 - \chi^2) = 0$

Generically, $\xi_\pm$ are not constant and depend on values of $\tau$ and $\chi$.




Singularity is not cured upon including Galileon terms.

# k-essence as the model of two scalar fields

$$S = \int d^4x \left[ \frac{\lambda^2}{2} (\partial_\mu \varphi)^2 - V(\lambda) \right] .$$

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$$\frac{V'(\lambda)}{\lambda} = (\partial_\mu \varphi)^2 \equiv X .$$

$V(\lambda) \propto -\lambda^2 + \lambda^4 \Longrightarrow \mathcal{L}(X) \propto X + X^2$ sub-luminal

$V(\lambda) \propto \lambda^2 - \lambda^4 \Longrightarrow \mathcal{L}(X) \propto X - X^2$ super-luminal

# More than just k-essence!

$$V(\lambda) = \frac{\lambda^2}{2} \Longrightarrow \lambda \text{ is a Lagrange multiplier}$$

$$S = \int d^4x \frac{\lambda^2}{2} \left[ (\partial_\mu \varphi)^2 - 1 \right] \Longrightarrow T_{\mu\nu} = \lambda^2 \partial_\mu \varphi \partial_\nu \varphi \; .$$

Pressureless perfect fluid!

$$V(\lambda) = \frac{\lambda^2}{2} + \frac{\lambda^4}{4} \Longrightarrow \mathcal{L}(X) = (X-1)^2$$

Ghost condensate develops caustic singularities Arkani-Hamed et *al*'05.

Idea: promote $\lambda$ to a dynamical degree of freedom.

$$S = \int d^4x \left[ \frac{(\partial_\mu \lambda)^2}{2M^2} + \frac{\lambda^2}{2}(\partial_\mu \varphi)^2 - V(\lambda) \right] \qquad M \text{ is large!!! .}$$

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Field transformation

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Field transformation

$$\tilde{\lambda} = \frac{\lambda}{M} \qquad \tilde{\varphi} = M\varphi$$

$$\Psi = \tilde{\lambda} e^{i\tilde{\varphi}}$$

$$S = \int d^4x \left[ \frac{1}{2} \partial_\mu \Psi^* \partial^\mu \Psi - V(|\Psi|) \right] .$$

The model is manifestly caustic free and exhibits luminal propagation.

Consider the simple renormalizable potential,

$$V(\Psi) = \frac{\alpha \cdot M^2 \cdot |\Psi|^2}{2} + \frac{\beta \cdot M^4 \cdot |\Psi|^4}{4 \cdot \Lambda^4}$$

- $\alpha > 0$, $\beta = 0 \Longrightarrow$ pressureless perfect fluid.

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- $\alpha > 0$ and $\beta > 0 \Longrightarrow$ ghost condensate $\mathcal{L}(X) \propto (X-1)^2$

  Arkani-Hamed et al'03 '05

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Strategy: pick the simplest possible case $V(\Psi) = \frac{M^2|\Psi|^2}{2}$
and derive main conclusions out of it.

# Degree of freedom counting: 2 vs.1

General solution

$$\Psi = \int d\mathbf{k}\alpha(\mathbf{k})e^{i\mathbf{kx}+i\sqrt{k^2+M^2}t} + \int d\mathbf{k}\beta(\mathbf{k})e^{-i\mathbf{kx}-i\sqrt{k^2+M^2}t}$$

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Only the branch with fixed frequency reproduces pressureless perfect fluid,

$$\boxed{\beta(\mathbf{k}) = 0} \Longrightarrow \Psi = \int d\mathbf{k}\alpha(\mathbf{k})e^{i\mathbf{kx}+i\sqrt{k^2+M^2}t}$$

Provided that $\boxed{L^{-1} \ll M \cdot v}$

$$\frac{\partial\delta\tilde{\varphi}}{\partial t} = \sqrt{M^2 + (\partial_i\tilde{\varphi})} \Longrightarrow \partial_\mu\varphi\partial^\mu\varphi = 1 \Longrightarrow$$

$$\frac{du^\mu}{ds} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta = 0 \qquad \partial_\mu\varphi = u_\mu$$

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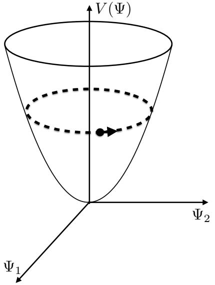


$$\Psi = Ae^{iMt} + Be^{-iMt} \to B = 0 \to \Psi = Ae^{iMt}$$

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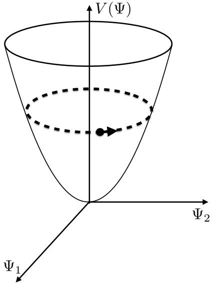


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Cosmic drag: $$\Psi = \frac{A}{a^{3/2}} e^{iMt} \Longrightarrow \mathcal{P} = 0, \ \rho \propto \frac{1}{a^3}$$

$$\Psi = Ae^{iMt} \Longrightarrow \Psi = \int d\mathbf{k}\alpha(\mathbf{k})e^{i\mathbf{kx}+i\sqrt{k^2+M^2}t}$$

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Madelung transformation: $$\tilde{\Psi} = \sqrt{\rho}\cdot e^{i\tilde{\varphi}-iMt} \ .$$

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$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \qquad \mathbf{v} = -\frac{\nabla\tilde{\varphi}}{M}$$

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\frac{1}{2M^2}\nabla\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}$$ "Quantum pressure"

# Using Schroedinger equation in cosmology

- Trick to avoid cumbersome N-body simulations,

  Widrow and Kaiser'91   C. Uhlemann et *al*'14

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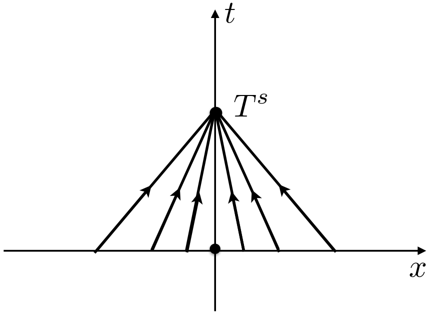
- Physically relevant situation: super-light axion in the Bose-Einstein state, $m \sim 10^{-22}$ eV.

  Sin'94 Ji and Sin'94 Lee and Koh'96 Hu et *al*'00

# Perfect caustics



# Resolving caustic singularity

$$\rho(t=0) = A^2 \exp\left(-\frac{x^2}{L^2}\right) \quad v(t=0) = -\beta x \quad \text{Hui et } al'16$$

$$\Longrightarrow v = -\frac{x}{T^s - t} \quad T^s = \frac{1}{\beta}$$

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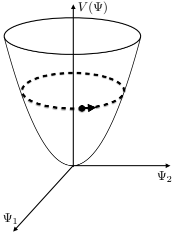
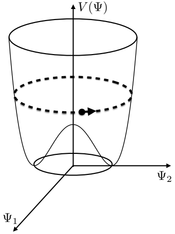
$$\boxed{T^s \to T = T_1 - iT_2} \qquad \frac{T_2}{T_1} \propto \frac{1}{M}$$

Collapse time is promoted to complex number $\Longrightarrow$ singularity is not developed in the real time.

$$v = -\frac{x}{T^s - t} \to v = -\frac{x(T_1 - t)}{(T_1 - t)^2 + T_2^2}$$

# k-essence



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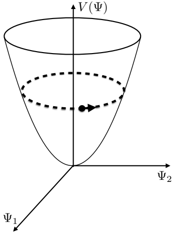


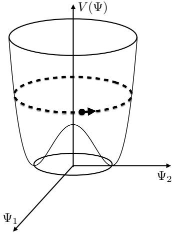


$$\Psi = \frac{A}{a^{3/2}} e^{iMt} \to \Psi = \frac{A}{a^{3/2}\sqrt{2\omega}} e^{\pm i \int dt \omega} \qquad \omega = \sqrt{\frac{2\partial V}{\partial |\Psi|^2}}$$

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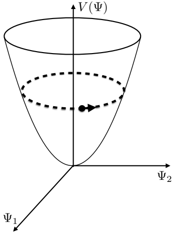


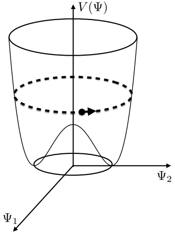


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$$2\sqrt{\frac{2\partial V}{\partial |\Psi|^2}} |\Psi|^2 = \frac{A^2}{a^3} \qquad V(\Psi) \propto |\Psi|^4 \Longrightarrow \mathcal{P} = \frac{1}{3}\rho \qquad (\text{cf. } \mathcal{L} \propto X^2)$$

# Conclusions

- k-essence, pressureless perfect fluid and ghost condensate belong to the same class of models with two scalar fields.
- Simple caustic free completion by means of the complex scalar designed.
- Only specific initial conditions allow to reproduce pressureless perfect fluid or k-essence.
- Mechanism of resolving caustic singularities: the collapse time is promoted to the complex number in the complete picture.

Thanks for your attention!