Hawking radiation of black hole with supertranslation field

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Schwarzschild metric in isotropic spherical coordinates and its generalization with supertranslation field

$$ds^2 = -\frac{(1-M/2\rho)^2}{(1+M/2\rho)^2}dt^2 + (1+M/2\rho)^4(d\rho^2 + \rho^2 d\Omega^2)$$

Generalization of the Schwarzschild metric with supertranslation field (G.Compere and J.Long, 2016)

$$ds^2 = -\frac{(1-M/2\rho_s)^2}{(1+M/2\rho_s)^2}dt^2 + (1+M/2\rho_s)^4 \times$$

$$\left(d\rho^2 + (((\rho-E)^2 + U)\gamma_{AB} + (\rho-E)C_{AB})dz^A dz^B\right)$$

$$\rho_s(\rho, C) = \sqrt{(\rho - C)^2 + D_A C D^A C}$$

$$\gamma_{AB} dz^A dz^B = d\theta^2 + \sin^2\theta d\varphi^2$$

$$C_{AB} = -(2D_A D_B - \gamma_{AB} D^2)C$$

Supertranslation field

$$\mathbf{C} = \mathbf{C}(\theta)$$

$$C_{\theta\theta} = -\left(C'' - C' \cot\theta\right), \qquad C_{\varphi\varphi} = -\sin^2\theta \left(C'' - C' \cot\theta\right)$$

$$C_{\varphi\theta} = 0$$

Kerr case: $C_{\theta\theta}(\theta) = a/\sin\theta, C(\theta) = a_2 + a_1 \cos\theta + a \sin\theta$ **(S.J.Fletcher and A.W.Lun, 2003, G.Barnich and C.Troessaert, 2011)**

Notations

$$L := 1 - \frac{2M}{r}$$

$$K := r - M + rL^{1/2}$$

Transformation from isotropic to Schwarzschild variables $\mathbf{r} = \mathbf{r}(\rho_s(\rho, \mathbf{C})) \Rightarrow \rho = \rho(\mathbf{r}, \mathbf{C})$

$$\rho_s(\rho, C) = \sqrt{(\rho - C)^2 + D_A C D^A C}$$

$$r := \rho_s(\rho, C)\left(1 + \frac{M}{2\rho_s(\rho, C)}\right)^2$$

$$b := \frac{2C'}{K}$$

$$\rho = C + \frac{K}{2}\left(1 - \frac{4(DC)^2}{K^2}\right)^{1/2} = C + \frac{K}{2}\sqrt{1 - b^2}$$

$$d\rho = \frac{K}{2}\left[\left(b - \frac{bb'}{\sqrt{1 - b^2}}\right)d\theta + \frac{dr}{rL^{1/2}\sqrt{1 - b^2}}\right].$$

Horizon

$$\rho_s(\rho_H, C) = M/2 \quad \Rightarrow \quad r = 2M$$

Transformed metric with supertranslation field in Schwarzschild variables

$$ds^2 = -Ldt^2 + \frac{dr^2}{L(1-b^2)} + 2drd\theta \frac{b(\sqrt{1-b^2}-b')r}{(1-b^2)L^{1/2}}$$
$$+d\theta^2 r^2 \frac{(\sqrt{1-b^2}-b')^2}{1-b^2} + d\varphi^2 r^2 \sin^2\theta (b\cot\theta - \sqrt{1-b^2})^2.$$

Inverse metric (t, r, θ) **part**

$$\begin{pmatrix} L^{-1} & 0 & 0 \\ 0 & L & -L^{1/2} b[r\sqrt{1-b^2}-b']^{-1} \\ 0 & -L^{1/2} b[r\sqrt{1-b^2}-b']^{-1} & [r(\sqrt{1-b^2}-b')]^{-2} \end{pmatrix}$$

Equation for eigenmodes

$$\Box\psi = \frac{1}{\sqrt{|g|}}\partial_\mu\sqrt{|g|}g^{\mu\nu}\partial_\nu\psi = 0$$

Perturbative solution of the equation in parameter $\frac{O(C)}{K} \sim \frac{O(C)}{M}$ **in the near-horizon region** $L = 1 - 2M/r \ll 1$

$$\psi = \psi_0 + \psi_1 + \psi_2 + \cdots$$

$$\Box\psi = \Box_0\psi_0 + \Box_1\psi_0 + \Box_0\psi_1 + \Box_2\psi_0 + \Box_0\psi_2 + \Box_1\psi_1 + \cdots .$$

Subscripts denote the orders in $\frac{O(C)}{K}$.

S-wave mode

Zero order equation

$$\Box_0\psi = (rL)^{-1}\left[-\partial_t^2 + \partial_{r_*}^2 + L\left(\frac{2M}{r^3} - \frac{\hat{K}^2(\theta,\varphi)}{r^2}\right)\right] r\psi$$

First order solution ψ_1

Zero-order S-mode

$$\psi_0 = \frac{e^{i\omega(t\pm r_*)}}{\sqrt{4\pi}r}$$

The first order equation $\Box_1\psi_0 + \Box_0\psi_1 = 0$

$$\Box_1\psi_0 = L\left(\frac{1}{\sqrt{-g}}\partial_r\sqrt{-g}\right)\Bigg|_{(1)}\partial_r\psi_0 + \frac{1}{\sqrt{-g}}\partial_\theta(\sqrt{-g}\,g_1^{r\theta})\,\partial_r\psi =$$

$$= \left[L\partial_r(-b' - b\cot\theta) - \left(\frac{L^{1/2}}{r}\right)(b' + b\cot\theta)\right]\partial_r\psi_0 = 0$$

$$L\partial_r b = L\partial_K(2C/K)\partial_r K = -bL^{1/2}/r$$

$$\psi_1 = \psi_0$$

Second-order equation $\Box_2\psi_0 + \Box_0\psi_2 = 0$

$$\Box_2\psi_0 = \left[L \left(\frac{\partial_r\sqrt{-g}}{\sqrt{-g}} \right)_{(2)} + \left(\frac{\partial_\theta\sqrt{-g}}{\sqrt{-g}} \right) g^{r\theta} \Big|_{(2)} + \partial_\theta g^{r\theta}|_{(2)} \right] \partial_r\psi_0$$

Derivatives ∂_r produce powers L^{-1}

$$\partial_r \frac{e^{i\omega r_*}}{r} \simeq \frac{e^{i\omega r_*}}{r} \frac{i\omega}{L}, \qquad \partial_r^2 \frac{e^{i\omega r_*}}{r} \simeq -\left(\frac{\omega^2}{L^2} + \frac{2iM\omega}{L^2\, r^2} \right) \frac{e^{i\omega r_*}}{r}.$$

In the leading order in L^{-1}

$$\Box_2\psi_0 = \frac{F_{(2)}(\theta)}{K^2} \frac{\pm i\omega}{L^{1/2}} \psi_0$$

Solution of the eqution $\Box_2\psi_0 + \Box_0\psi_2 = 0$

$$\frac{1}{rL}\left[-\partial_t^2 + \partial_{r_*}^2 + L\left(\frac{2M}{r^3} - \frac{\hat{K}^2(\theta,\varphi)}{r^2}\right)\right] r\psi_2 = \frac{\pm i\omega}{L^{1/2}}\frac{F_{(2)}}{K^2}\frac{e^{i\omega(t\pm r_*)}}{r}$$

Looking for solution in the form $r\psi_2 = L^{1/2}\varphi\, e^{i\omega(t\pm r_*)}$, we obtain

$$\left[\left(\frac{M}{r^2}\right)^2 \pm 2i\omega\left(\frac{M}{r^2}\right)\right]\varphi = \pm i\omega\frac{F_{(2)}}{K^2}$$

Second-order solution

$$\psi_2 \simeq L^{1/2}\frac{\pm i\omega r^2}{(1 \pm i4r\omega)}\frac{F_{(2)}}{K^2}\psi_0\bigg|_{r\sim 2M}$$

The equation for ψ_n

$$\Box_n\psi_0 + \sum \Box_k\psi_l + \Box_0\psi_n = 0$$

$$\partial_r \frac{O(C^n)}{K^n} = -n\frac{O(C^n)}{K^n r L^{1/2}}.$$

Because

i) $g^{rr} = L$ does not have contributions of higher orders in $O(C)/M$, in equations of higher order there are no terms $\partial_r g^{rr}_{(n)}\partial_r$.

ii) ∂_θ does not change the order in L,

we obtain

Solution ψ_n

$$F_{(2)}/K^2 \to F_{(n)}/K^n$$

$$\psi_n \sim L^{1/2} O(C^n/K^n)\psi_0$$

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Isotropic geodesics (S.Chandrasekhar, 1983)

Lagrangian

$$2\mathcal{L} = -L\dot{t}^2 + \frac{\dot{r}^2}{L}\hat{g}_{rr} + 2\frac{r\dot{r}\dot{\theta}}{L^{1/2}}\hat{g}_{r\theta} + r^2\left[\dot{\theta}^2\hat{g}_{\theta\theta} + \dot{\varphi}^2\sin^2\theta\hat{g}_{\varphi\varphi}\right]$$

Equation for $r(\tau)$

$$\frac{\ddot{r}}{L}\hat{g}_{rr} - \frac{\dot{r}^2}{r^2L^2}\hat{g}_{rr} + \frac{E^2}{r^2L^2} + \frac{\dot{r}^2}{2L}\partial_r\hat{g}_{rr} + \frac{\dot{r}\dot{\theta}}{L}\partial_\theta\hat{g}_{rr} + \frac{\ddot{\theta}r}{L^{1/2}}\hat{g}_{r\theta}$$
$$+\dot{\theta}^2\left(\frac{r}{L^{1/2}}\partial_\theta\hat{g}_{r\theta} - r\hat{g}_{\theta\theta} - \frac{r^2}{2}\partial_r\hat{g}_{\theta\theta}\right) - \dot{\varphi}^2\left(r\hat{g}_{\varphi\varphi} + \frac{r^2}{2}\partial_r\hat{g}_{\varphi\varphi}\right) = 0$$

Equation for $\theta(\tau)$

$$\ddot{\theta}r^2\hat{g}_{\theta\theta} + 2r\dot{\theta}\dot{r}\hat{g}_{\theta\theta} + r^2\dot{\theta}\dot{r}\partial_r\hat{g}_{\theta\theta} + \frac{1}{2}r^2\dot{\theta}^2\partial_\theta\hat{g}_{\theta\theta} + \frac{\ddot{r}r + \dot{r}^2}{L^{1/2}}\hat{g}_{r\theta} - \frac{\dot{r}^2}{rL^{3/2}}\hat{g}_{r\theta} +$$
$$+\frac{\dot{r}^2 r}{L^{1/2}}\partial_r\hat{g}_{r\theta} - \frac{\dot{r}^2}{2L}\partial_\theta\hat{g}_{rr} - \dot{\varphi}^2 r^2(\sin\theta\cos\theta\hat{g}_{\varphi\varphi} + \sin^2\theta\partial_\theta\hat{g}_{\varphi\varphi}) = 0.$$

Ansatz for solution for radial null geodesic in the near-horizon region

Ansatz

$$\dot{t} = E/L,$$
$$\dot{r} = C + C_1 L^{1/2} + \cdots ,$$
$$\dot{\theta} = A\, L^{-1/2} + A_1 + \cdots ,$$
$$\varphi = const$$

Leading in L^{-1} parts of equations

$$L^{-2}\left[E^2 - C^2\hat{g}_{rr} - rAC\hat{g}_{r\theta}\right] = 0,$$
$$L^{-3/2}\left[C\hat{g}_{r\theta} + rA\hat{g}_{\theta\theta}\right] = 0.$$

Solution for C and A

$$C^2 = E^2$$
$$A \simeq -\frac{C\hat{g}_{r\theta}}{2M\hat{g}_{\theta\theta}}\bigg|_{L\ll 1}$$

Leading-order solution for geodesic

Equation $\mathcal{L} = 0$

$$L^{-1}[C^2\hat{g}_{rr} - E^2 + 2rCA\hat{g}_{r\theta} + r^2A^2\hat{g}_{\theta\theta}] \equiv 0.$$

$$\hat{g}_{rr}\hat{g}_{\theta\theta} - \hat{g}_{r\theta}^2 = \hat{g}_{\theta\theta}$$

Solution for geodesics $L \ll 1$

$$t = t_0 + E\tau$$
$$r = 2M + E\tau$$
$$\theta = \frac{\pi}{2} - \frac{\hat{g}_{r\theta}}{2\hat{g}_{\theta\theta}}L^{1/2}.$$

Schwarzschild case

$$t = t_0 + E\tau$$
$$r = 2M + E\tau$$
$$\theta = \frac{\pi}{2}$$

Conclusion

In the near-horizon region, relevant for calculation of the Hawking radiation, both solutions for the eigenmodes and isotropic geodesics, calculated in Schwarzschild geometry with supertranslation field, have the form similar to solutions in the pure Schwarzschild case.

The accuracy of the above calculations in $L = 1 - 2M/r$ is the same as in the "standard" calculations (Hawking, 1975, N.D.Birrel and P.C. Davies, 1982, Brout et al.,1995,...).

With the same accuracy, the density matrix of radiation is the same as in the Schwarzschild case.

Thank you