# Gauged spinning multiparticle models with the N=4 deformed supersymmetry

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based on

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In this talk there will be presented new models of the $\mathcal{N}=4$ deformed supersymmetric mechanics with $SU(2|1)$ symmetry based on gauging the systems with dynamical $(\mathbf{1},\mathbf{4},\mathbf{3})$ multiplets and semi-dynamical $(\mathbf{4},\mathbf{4},\mathbf{0})$ ones.

The models of the deformed superymmetric mechanics with $SU(2|1)$ symmetry have been considered in recent studies [E. Ivanov, S. Sidorov, 2013, 2015].
Such supersymmetry is an extension of simple supersymmetry that alternative to higher $\mathcal{N}$-extension of the Poincaré superalgebras.

The models with $SU(2|1)$ symmetry have been considered earlier in [A.V. Smilga; S. Bellucci, A. Nersessian, 2004] where such symmetry was named as "weak supersymmetry".

Recently, there is a request to study systems with curved rigid supersymmetry using supergroup $SU(2|1)$ and its central extension [T.T. Dumitrescu, G. Festuccia, N. Seiberg, 2012; I.B. Samsonov, D. Sorokin, 2014]

The centrally-extended superalgebra $\hat{su}(2|1)$ is defined by the anticommutators $(i=1,2)$

$$\{Q^i,\bar{Q}_k\}=2mI^i_k+2\delta^i_k\,(H-mF)\,,\qquad \{Q^i,Q^k\}=\{\bar{Q}_i,\bar{Q}_k\}=0$$

of the odd generators $Q^i$ and $\bar{Q}_i=(Q^i)^\dagger$. The generator $H=H^\dagger$ commutes with all other generators. The $SU(2)_{\rm int}$ generators $I^i_k=(I^k_i)^\dagger$ and the $U(1)_{\rm int}$ generator $F=F^\dagger$,

$$[I^i_j,I^k_l]=\delta^k_j I^i_l-\delta^i_l I^k_j\,,\qquad [I^i_j,F]=0\,,$$

have nonvanishing commutators with supercharges

$$[I^i_j,Q^k]=\delta^k_j Q^i-\tfrac{1}{2}\,\delta^i_j Q^k\,,\qquad [F,Q^k]=\tfrac{1}{2}\,Q^k\,,\quad [F,\bar{Q}_l]=-\tfrac{1}{2}\,\bar{Q}_l\,.$$

$m$ is the mass parameter; the limit $m=0$ leads to the $N=4$, $d=1$ Poincaré superalgebra.

Also, there is the automorphism group $SU(2)_{\rm ext}$ with the generators $T^i_j=(T^k_i)^\dagger$ which rotate the charges in the precisely same way as the internal $SU(2)_{\rm int}$ generators $I^i_j$ do:

$$[T^i_j,Q^k]=\delta^k_j Q^i-\tfrac{1}{2}\,\delta^i_j Q^k\,,\qquad [T^i_j,I^k_l]=\delta^k_j I^i_l-\delta^i_l I^k_j\,.$$

In [E. Ivanov, S. Sidorov, 2013] there were constructed $SU(2|1)$ supersymmetry invariant one-particle models in the superspace with coordinates $\left(t, \theta_k, \bar\theta^k\right)$, $\bar\theta^i = (\overline{\theta_i})$, parametrized the coset with representatives $\exp\left\{itH + \vartheta_k Q^k + \bar\vartheta^k \bar Q_k\right\}$ where $\vartheta_i = (1 + \frac{2}{3}\, m\, \theta_k \bar\theta^k)\theta_i$, $\bar\vartheta^i = (1 + \frac{2}{3}\, m\, \theta_k \bar\theta^k)\bar\theta^i$. The odd $SU(2|1)$ transformations of these coordinates are

$$\delta t = i(\epsilon_k\, \bar\theta^k + \bar\epsilon^k\, \theta_k)\,, \qquad \delta\theta_i = \epsilon_i + 2m\, \bar\epsilon^k\, \theta_k\, \theta_i\,, \qquad \delta\bar\theta^i = \bar\epsilon^i - 2m\, \epsilon_k\, \bar\theta^k\, \bar\theta^i\,.$$

As a further step, in [E. Ivanov, S. Sidorov, 2015] there was considered the “minimal” complex harmonic coset

$$\frac{\{H, Q^\pm, \bar Q^\pm, F, I^{\pm\pm}, I^0, T^{\pm\pm}, T^0\}}{\{F, I^{++}, I^0, I^{--} - T^{--}, T^0\}} \sim \left(t_A, \theta^\pm, \bar\theta^\pm, w_i^\pm\right) \equiv \zeta_H,$$

$$\text{where} \qquad Q^+ \equiv Q^1\,, \qquad Q^- \equiv Q^2\,, \qquad \bar Q^- \equiv \bar Q_1\,, \qquad \bar Q^+ \equiv -\bar Q_2\,.$$

$$I^{++} \equiv I_2^1\,, \quad I^{--} \equiv I_1^2\,, \quad I^0 \equiv I_1^1 - I_2^2 = 2I_1^1\,, \qquad T^{++} \equiv T_2^1\,, \quad T^{--} \equiv T_1^2\,, \quad T^0 \equiv T_1^1 - T_2^2 = 2T_1^1\,.$$

The use of harmonics $w_i^\pm$ given additional opportunities to build new physical models, but it is remain a number of important outstanding issues.

As it was indicated in [E. Ivanov, S. Sidorov, 2013, 2015] that there are still awaited the $SU(2|1)$ generalizations of the $\mathcal{N} = 4$ supersymmetric Calogero-like systems, the gauging procedure and coupling to the background gauge fields.

From the point of view of the $\mathcal{N} = 4$ mechanics, all of these issues are interrelated.

- The WZ terms in the particle action describe the interaction with external gauge fields [E. Ivanov, O. Lechtenfeld, 2003].
- The actions of the same type describe semi-dynamical degrees of freedom [SF, E. Ivanov, O. Lechtenfeld, 2009, 2010], the use of which is important in the construction of many-particle systems [SF, E. Ivanov, O. Lechtenfeld, 2008, 2012].
- An additional ingredient in this design is the introduction of pure gauge degrees of freedom and using the gauging procedure [F. Delduc, E. Ivanov, 2006].

Here we will present new models of the $\mathcal{N}=4$ deformed supersymmetric mechanics that make use of a few different types of $SU(2|1)$ supermultiplets:
dynamical, semi-dynamical and pure gauge supermultiplets.
As result of it we will obtain new $SU(2|1)$-invariant one-particle model with spinning degrees of freedom, as well as new $SU(2|1)$ superextension of the Calogero-Moser multi-particle system.

The harmonic superspace [E. Ivanov, S. Sidorov, 2015] is not directly applicable for solving these tasks.

The main problem roots in the algebra of the covariant constraints to be imposed on the relevant harmonic superfields $\Psi$ for singling out various irreducible $SU(2|1)$ multiplets. The Grassmann analyticity conditions in the harmonic superspace [E. Ivanov, S. Sidorov, 2015] (specifically, $\mathcal{D}^{+}\Psi=0$, $\bar{\mathcal{D}}^{+}\Psi=0$)
necessarily entail the harmonic condition (specifically, $\mathcal{D}^{++}\Psi=0$).

However, such harmonic constraints turn out to be too strong if we wish to describe some supermultiplets in the harmonic approach, e.g. the "topological" gauge multiplet which is the main object of the $d=1$ gauging [F. Delduc, E. Ivanov, 2006]
efficiently exploited in [SF, E. Ivanov, O. Lechtenfeld, 2008, 2009, 2010, 2012].

The only way around is to pass to an extended $SU(2|1)$ harmonic superspace involving two sets of harmonic variables: those associated with the group $SU(2)_{\rm int}$ and those parametrizing the external automorphism group $SU(2)_{\rm ext}$.

Now we introduce new harmonic superspace,
including the standard (unitary) harmonics on $SU(2)_{\rm ext}$.
As a result, we gain an opportunity to perform a gauging procedure and define interacting dynamical and semi-dynamical multiplets.

# SU(2|1) harmonic superspace revisited

As opposed to the "minimal" harmonic coset [E. Ivanov, S. Sidorov, 2015], we will use now

$$\hat{\zeta}_H = \left(t_A, \theta^\pm, \bar{\theta}^\pm, u_i^\pm, z^{++}\right) \quad \sim \quad \frac{\{H, Q^\pm, \bar{Q}^\pm, F, I^{\pm\pm}, I^0, T^{\pm\pm}, T^0\}}{\{F, I^{++}, I^0, T^0\}}\,,$$

$$\text{where} \qquad u_i^\pm\,, \qquad u^{+i}u_i^- = 1\,, \qquad u_i^+u_k^- - u_k^+u_i^- = \varepsilon_{ik}$$

are the standard unitary harmonics on the coset $SU(2)_{\rm ext}/U(1)_{\rm ext} \sim S^2$ [GIKOS, 1984], while the coordinate $z^{++}$ is associated with the generator $I^{--}$.

The elements of this coset are defined as

$$g_H = e^{i\left(\xi T^{++} + \bar{\xi} T^{--}\right)} \exp\left\{z^{++}I^{--}\right\} \exp\left\{it_A H - \theta^+ Q^- + \bar{\theta}^+\bar{Q}^-\right\} \exp\left\{\theta^- Q^+ - \bar{\theta}^-\bar{Q}^+\right\}\,,$$

where $e^{i(\,\xi\tau^{++} + \bar{\xi}\tau^{--})} = (u_i^\pm)$.

Non-unitary harmonics $w_i^\pm$ are related to the standard harmonics $u_i^\pm$ as [A. Galperin, E. Ivanov, O. Ogievetsky, 1994]

$$w_i^+ = u_i^+ + z^{++}u_i^-\,, \qquad w_i^- = u_i^-\,, \qquad w_i^+w_k^- - w_k^+w_i^- = \varepsilon_{ik}\,.$$

The odd $SU(2|1)$ transformations are written as

$$\delta t_A = 2i\left(\epsilon^-\bar{\theta}^+ - \bar{\epsilon}^-\theta^+\right)\,,$$

$$\delta\theta^+ = \epsilon^+ + \epsilon^-\left(z^{++} - m\,\theta^+\bar{\theta}^+\right)\,, \qquad \delta\theta^- = \epsilon^- + 2m\,\bar{\epsilon}^-\theta^-\theta^+,$$

$$\delta z^{++} = m\left(\epsilon^+\bar{\theta}^+ + \bar{\epsilon}^+\theta^+\right) + m\,z^{++}\left(\epsilon^-\bar{\theta}^+ + \bar{\epsilon}^-\theta^+\right)\,, \qquad \delta u_i^\pm = 0\,,$$

where $\epsilon^\pm = \epsilon^i u_i^\pm$, $\bar{\epsilon}^\pm = \bar{\epsilon}^k u_k^\pm$. It follows that the $SU(2|1)$ harmonic superspace contains the analytic harmonic subspace parametrized by the reduced coordinate set

$$\hat{\zeta}_A = (t_A, \theta^+, \bar{\theta}^+, u_i^\pm, z^{++})\,.$$

which is closed under the action of $SU(2|1)$.

In standard way we derive the covariant derivatives, in particular, half of the fermionic ones are short

$$\mathcal{D}^+ = \frac{\partial}{\partial \theta^-} - m\,\bar\theta^- \tilde I^{++}\,, \qquad \bar{\mathcal{D}}^+ = -\frac{\partial}{\partial \bar\theta^-} + m\,\theta^- \tilde I^{++}\,,$$

part of the harmonic covariant derivatives is

$$\mathcal{D}^0 \,=\, \partial^0_u + 2z^{++}\frac{\partial}{\partial z^{++}} + \left(\theta^+\frac{\partial}{\partial \theta^+} + \bar\theta^+\frac{\partial}{\partial \bar\theta^+}\right) - \left(\theta^-\frac{\partial}{\partial \theta^-} + \bar\theta^-\frac{\partial}{\partial \bar\theta^-}\right),$$

$$\mathcal{D}^{++} = \partial^{++}_u + z^{++}\left(\mathcal{D}^0 + \tilde I^0\right) + \ldots .$$

where $\tilde F$, $\tilde I^0$, $\tilde I^{++}$ are matrix parts of the generators $F$, $I^0$, $I^{++}$ properly acting on the matrix indices of the superfields and the operators.

The harmonic superfields $\Psi^{(q)}(t_A, \theta^\pm, \bar\theta^\pm, u^\pm, z^{++})$ are eigenfunctions of the harmonic $U(1)$ charge operator $\mathcal{D}^0$:

$$\mathcal{D}^0 \Psi^{(q)} = q\Psi^{(q)}\,.$$

We assume the polynomial dependence on $z^{++}$ and the standard harmonic expansion in $u^\pm$.

We limit our study to the harmonic superfields subjected to additional covariant conditions

$$\left(\mathcal{D}^0 + \tilde I^0\right)\Psi^{(q)} = 0 \quad \Rightarrow \quad \tilde I^0 \Psi^{(q)} = -q\Psi^{(q)}\,,$$

$$\tilde F\,\Psi^{(q)} = 0\,, \qquad \tilde I^{++}\Psi^{(q)} = 0\,.$$

The constraint $\tilde{I}^{++}\Psi^{(q)} = 0$ is the self-consistency condition for the covariant definition of the analytic $SU(2|1)$ superfields which live on the analytic subspace and satisfy the Grassmann-analyticity constraints

$$\mathcal{D}^{+}\Psi^{(q)} = \bar{\mathcal{D}}^{+}\Psi^{(q)} = 0\,,$$

which, due to the relation $\{\mathcal{D}^{+}, \bar{\mathcal{D}}^{+}\} = 2m\tilde{I}^{++}$, necessarily imply $\tilde{I}^{++}\Psi^{(q)} = 0$.

It is important, that as opposed to the approach of [E. Ivanov, S. Sidorov, 2015], these fermionic constraints by no means require the condition $\mathcal{D}^{++}\Psi^{(q)} = 0$, which is unnecessary to define some multiplets.

The notable property of the covariant derivatives is

$$\mathcal{D}_{z}^{--} - \mathcal{D}^{--} = \frac{\partial}{\partial z^{++}} - \partial_{u}^{--}$$

and we consider the harmonic superfields subjected to additional covariant condition

$$\left(\mathcal{D}_{z}^{--} - \mathcal{D}^{--}\right)\Psi^{(q)} = 0\,.$$

This constraint effectively eliminates the dependence of the harmonic superfields on the variable $z^{++}$

$$\Psi^{(q)}(t_A, \theta^{\pm}, \bar{\theta}^{\pm}, u^{\pm}, z^{++}) = e^{\,z^{++}\partial_{u}^{--}}\,\Phi^{(q)}(t_A, \theta^{\pm}, \bar{\theta}^{\pm}, u^{\pm}) = \Phi^{(q)}(t_A, \theta^{\pm}, \bar{\theta}^{\pm}, w^{\pm})\,.$$

Therefore, such harmonic superfields depend on $w_i^{+} = u_i^{+} + z^{++}u_i^{-}$, $w_i^{-} = u_i^{-}$ and we obtain the link with the models, which were considered in [E. Ivanov, S. Sidorov, 2015].

Of course, the analytic harmonic superfields are expressed as

$$\Psi^{(q)}(t_A, \theta^{+}, \bar{\theta}^{+}, u^{\pm}, z^{++}) = e^{\,z^{++}\partial_{u}^{--}}\,\Phi^{(q)}(t_A, \theta^{+}, \bar{\theta}^{+}, u^{\pm}) = \Phi^{(q)}(t_A, \theta^{+}, \bar{\theta}^{+}, w^{\pm})\,.$$

# The multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$

The multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ is described by even real superfield $\mathcal{X}$ subjected to the conditions

$$\mathcal{D}^0\,\mathcal{X}=0\,,\qquad \mathcal{D}^{++}\,\mathcal{X}=0\,,\qquad (\mathcal{D}_z^{--}-\mathcal{D}^{--})\mathcal{X}=0\,,\qquad \tilde{I}^0\,\mathcal{X}=\tilde{F}\,\mathcal{X}=\tilde{I}^{++}\,\mathcal{X}=0\,,$$

$$\mathcal{D}^-\mathcal{D}^+\,\mathcal{X}=0\,,\qquad \bar{\mathcal{D}}^-\bar{\mathcal{D}}^+\,\mathcal{X}=0\,,\qquad \left(\mathcal{D}^-\bar{\mathcal{D}}^+ + \bar{\mathcal{D}}^-\mathcal{D}^+\right)\mathcal{X}=2m\mathcal{X}\,,$$

These constraints are solved by

$$\mathcal{X}=x+\theta^-\psi^+ + \bar\theta^-\bar\psi^+ - \theta^+\psi^- - \bar\theta^+\bar\psi^- + \theta^-\bar\theta^- N^{++} + \theta^+\bar\theta^+ N^{--} + \ldots$$

Here, $N^{\pm\pm}=N^{ik}w_i^{\pm}w_k^{\pm}$, $\psi^{\pm}=\psi^i w_i^{\pm}$, and $x(t)$, $N^{ik}=N^{(ik)}(t)$, $\psi^i(t)$ are $d{=}1$ fields.

Free $\mathcal{X}$-action $S_{\mathcal{X}}=-\frac{1}{4}\int d\zeta_H\,\mathcal{X}^2$ yields the component action [A.V.Smilga, 2004]

$$S_{\mathcal{X}}=\frac{1}{2}\int dt\,\left[\dot{x}\dot{x}+i\left(\bar\psi_k\dot\psi^k-\dot{\bar\psi}_k\psi^k\right)-m^2x^2+2m\,\bar\psi_k\psi^k-\frac{1}{2}\,N^{ik}N_{ik}\right].$$

Another description of $(\mathbf{1},\mathbf{4},\mathbf{3})$ is through an analytic real prepotential $\mathcal{V}(\zeta_A)$ which is related to the superfield $\mathcal{X}$ (in the central basis) by the harmonic integral transform

$$\mathcal{X}(t,\theta_i,\bar\theta^i)=\int dw\left(1+m\,\theta^+\bar\theta^- - m\,\theta^-\bar\theta^+\right)^{-1}\mathcal{V}\left(t_A,\theta^+,\bar\theta^+,w^{\pm}\right)\Big|\,,$$

where the vertical bar $\Big|$ means that the expressions $t_A=t+i(\theta^+\bar\theta^- + \theta^-\bar\theta^+)$, $\theta^-=\theta^i w_i^-$, $\theta^+=\theta^i w_i^+(1+m\,\theta^k w_k^-\bar\theta^l w_l^+)$ should be substituted into the integrand.
This representation generalizes the analogous transform in the “flat” non-deformed $\mathcal{N}{=}4$ supersymmetric mechanics [F. Delduc, E. Ivanov, 2006; SF, E. Ivanov, O. Lechtenfeld, 2008].

In the WZ gauge for pregauge freedom $\delta\mathcal{V}=\mathcal{D}^{++}\lambda^{--}$, $\lambda^{--}=\lambda^{--}(\zeta_A)$ the fields appearing for $\mathcal{V}$ are identified with the fields in $\mathcal{X}$:

$$\mathcal{V}(\zeta_A)=x(t_A)-2\,\theta^+\psi^i(t_A)w_i^- - 2\,\bar\theta^+\bar\psi^i(t_A)w_i^- + 3\,\theta^+\bar\theta^+N^{ik}(t_A)w_i^-w_k^-\,.$$

# The multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ and SU(2|1) invariant WZ term

The multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ is described by the superfield $\mathcal{Z}^+(t_A,\theta^\pm,\bar\theta^\pm,z^{++},u^\pm)$ possessing the unit $U(1)$ charge, $\mathcal{D}^0\,\mathcal{Z}^+=\mathcal{Z}^+$, and satisfying the $SU(2|1)$ covariant constraints

$$(\mathcal{D}_z^{--}-\mathcal{D}^{--})\mathcal{Z}^+=0\,,\qquad \tilde{I}^0\,\mathcal{Z}^+=-\mathcal{Z}^+\,,\qquad \tilde{F}\,\mathcal{Z}^+=\tilde{I}^{++}\,\mathcal{Z}^+=0\,,$$
$$\mathcal{D}^{++}\,\mathcal{Z}^+=0\,,\qquad \mathcal{D}^+\mathcal{Z}^+=\bar{\mathcal{D}}^+\mathcal{Z}^+=0\,.$$

The general solution of the constraints is represented by the component expansion

$$\mathcal{Z}^+(t_A,\theta^+,\bar\theta^+,u^\pm,z^{++})=\mathscr{Z}^+(t_A,\theta^+,\bar\theta^+,w^\pm)=z^iw_i^+ +\theta^+\varphi+\bar\theta^+\phi-2i\theta^+\bar\theta^+\dot{z}^iw_i^-\,,$$

where $z^i(t_A)$, $\varphi(t_A)$, $\phi^i(t_A)$ are $d{=}1$ fields.

It has been shown in [E. Ivanov, S. Sidorov, 2015] that the WZ type actions enjoying $SU(2|1)$ supersymmetry cannot be constructed for the single multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$.

However, if we couple the multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ (superfield $\mathcal{Z}^+$) to the multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ (superfield $\mathcal{V}$) the $SU(2|1)$-invariant WZ action can be set up.

Such WZ action is given by the following integral over the analytic subspace

$$S_{\rm WZ}(\mathcal{V},\mathscr{Z}^+)=\tfrac{1}{2}\int d\zeta_A^{--}\,\mathcal{V}\,\mathscr{Z}^+\tilde{\mathscr{Z}}^+\,.$$

The corresponding component action $S_{\rm WZ}=\int dt\,L_{\rm WZ}$ with the component Lagrangian

$$L_{\rm WZ}=-\tfrac{i}{2}\,x\left(\bar{z}_k\dot{z}^k-\dot{\bar{z}}_kz^k\right)-\tfrac{1}{2}\,N^{kj}z_k\bar{z}_j+\tfrac{1}{2}\,\psi^k\left(z_k\bar\varphi+\bar{z}_k\phi\right)+\tfrac{1}{2}\,\bar\psi^k\left(z_k\bar\phi-\bar{z}_k\varphi\right)+\tfrac{1}{2}\,x\left(\varphi\bar\varphi+\phi\bar\phi\right)$$

is invariant under the $SU(2|1)$ transformations.

## Gauging of coupled dynamical $(\mathbf{1},\mathbf{4},\mathbf{3})$ and semi – dynamical $(\mathbf{4},\mathbf{4},\mathbf{0})$

The WZ action $S_{\rm WZ}$ is invariant with respect to the global $\mathrm{U}(1)$ transformations

$$\mathcal{Z}^{+\prime} = e^{i\lambda}\mathcal{Z}^{+}, \qquad \tilde{\mathcal{Z}}^{+\prime} = e^{-i\lambda}\tilde{\mathcal{Z}}^{+}\,.$$

Now we require local invariance of this action, with the parameter being promoted to an analytic superfield $\lambda = \lambda(\zeta_A)$

To provide this local symmetry in the considered system we introduce even analytic gauge superfield $V^{++}$, which satisfies the conditions

$$\mathcal{D}^{+}V^{++} = \bar{\mathcal{D}}^{+}V^{++} = 0\,, \qquad \tilde{I}^{++}V^{++} = 0\,,$$

$$(\mathcal{D}_z^{--} - \mathcal{D}^{--})V^{++} = 0\,, \qquad \mathcal{D}^{0}V^{++} = -\tilde{I}^{0}V^{++} = 2V^{++}\,, \qquad \tilde{F}V^{++} = 0$$

and is defined up to the gauge transformations

$$V^{++\prime} = V^{++} - D^{++}\lambda\,.$$

Using this $\mathrm{U}(1)$ gauge freedom we can choose the WZ gauge

$$V^{++} = 2i\,\theta^{+}\bar{\theta}^{+}A(t_A)\,.$$

The gauge superfield $V^{++}$ covariantizes the derivative $\mathcal{D}^{++}$. As a result, the complex analytic superfield $\mathcal{Z}^{+}$ is subject to the covariantized harmonic constraints

$$\nabla^{++}\,\mathcal{Z}^{+} \equiv (\mathcal{D}^{++} + i\,V^{++})\,\mathcal{Z}^{+} = 0\,.$$

The solution of this constraint in the WZ gauge is

$$\mathcal{Z}^{+}(t_A,\theta^{+},\bar{\theta}^{+},u^{\pm},z^{++}) = z^{i}w_i^{+} + \theta^{+}\varphi + \bar{\theta}^{+}\phi - 2i\,\theta^{+}\bar{\theta}^{+}\nabla_{t_A}z^{i}w_i^{-}\,,$$

where

$$\nabla z^{k} = \dot{z}^{k} + iA\,z^{k}\,, \qquad \nabla\bar{z}_k = \dot{\bar{z}}_k - iA\,\bar{z}_k\,.$$

We will consider the action

$$S = S_{\mathcal{X}} + S_{WZ} + S_{FI}\,,$$

where last term in the total action is the gauge-invariant Fayet-Iliopoulos (FI) term

$$S_{FI} = \tfrac{i}{2}\, c \int \mu_A^{(-2)}\, V^{++}\,,$$

which in the WZ gauge takes the form

$$S_{FI} = -c \int dt\, A\,.$$

After integrating over $\theta$s and harmonics and eliminating auxiliary fields the total action in the WZ gauge takes the following on-shell form (we make the redefinition $z^k \to z^k/\sqrt{x}$)

$$\begin{aligned}
S &= S_b + S_f\,,\\
S_b &= \frac{1}{2}\int dt\,\left[\dot{x}\dot{x} - m^2x^2 + i\left(\dot{\bar{z}}_k z^k - \bar{z}_k\dot{z}^k\right) - \frac{(\bar{z}_k z^k)^2}{4x^2} + 2A\left(\bar{z}_k z^k - c\right)\right],\\
S_f &= \int dt\left[\frac{i}{2}\left(\bar{\psi}_k\dot{\psi}^k - \dot{\bar{\psi}}_k\psi^k\right) + m\bar{\psi}_k\psi^k\right] - \int dt\,\frac{\psi^i\bar{\psi}^k z_{(i}\bar{z}_{k)}}{x^2}\,.
\end{aligned}$$

The last term in the bosonic action $S_b$ produces first class constraint $\bar{z}_k z^k - c \approx 0$ restricting the quantum spectrum to a single supermultiplet on fixed energy level.

The mass (frequency) of the physical states is defined by the deformation parameter of the $SU(2|1)$ supersymmetry.

# Matrix model

Now we are going to generalize one-particle model to the $\mathrm{U}(n)$, $d{=}1$ gauge theory following [SF, E. Ivanov, O. Lechtenfeld, 2008, 2012].

The matrix model to be constructed involves the following $\mathrm{U}(n)$ entities:

- $n^2$ commuting superfields

$$\mathcal{X}_b^a = (\widetilde{\mathcal{X}_a^b})\,, \qquad a,b = 1,\ldots,n$$

forming the hermitian $n{\times}n$-matrix superfield $\mathcal{X} = (\mathcal{X}_a^b)$
in adjoint representation of $\mathrm{U}(n)$;

- $n$ commuting complex superfields $\mathcal{Z}_a^+$ forming the $\mathrm{U}(n)$ spinor

$$\mathcal{Z}^+ = (\mathcal{Z}_a^+)\,, \qquad \tilde{\mathcal{Z}}^+ = (\tilde{\mathcal{Z}}^{+a})\,;$$

- $n^2$ non-propagating "gauge superfields"

$$V^{++} = (V^{++}{}_a^b)\,, \qquad (\widetilde{V^{++}{}_a^b}) = V^{++}{}_b^a\,.$$

The local $\mathrm{U}(n)$ transformations are given by

$$\mathcal{X}\,' = e^{i\lambda}\mathcal{X}e^{-i\lambda}, \qquad \mathcal{Z}^{+\,\prime} = e^{i\lambda}\mathcal{Z}^+, \qquad V^{++\,\prime} = e^{i\lambda}\,V^{++}\,e^{-i\lambda} - i\,e^{i\lambda}(D^{++}e^{-i\lambda}),$$

where $\lambda_a^b(\zeta_A) \in u(n)$ is the "hermitian" analytic matrix parameter, $\widetilde{\lambda} = \lambda$.

The $SU(2|1)$ matrix model with $\mathrm{U}(n)$ gauge symmetry is described by the action

$$S_{matrix} = S_{\mathcal{X}} + S_{WZ} + S_{FI}\,.$$

The first term in the total action, $S_{\mathcal{X}} = -\frac{1}{4}\int \mu_H \mathrm{Tr}\left(\mathcal{X}^{\,2}\right),$ is the gauged action of the $(\mathbf{1},\mathbf{4},\mathbf{3})$ multiplets. Now the superfields $\mathcal{X} = (\mathcal{X}_a^b)$ are subjected to covariantized constraints

$$\nabla^{++}\mathcal{X} = \mathcal{D}^{++}\mathcal{X} + i\,[V^{++},\mathcal{X}] = 0\,,$$

$$\nabla^{-}\nabla^{+}\mathcal{X} = 0\,, \qquad \bar{\nabla}^{-}\bar{\nabla}^{+}\mathcal{X} = 0\,, \qquad (\nabla^{-}\bar{\nabla}^{+} + \bar{\nabla}^{-}\nabla^{+})\,\mathcal{X} = 2m\mathcal{X}\,,$$

where the gauge connections in the spinor covariant derivatives are expressed through $V^{++}(\zeta,u)$.

The last term in the action is the FI term $S_{FI} = \frac{i}{2}\,c\int \mu_A^{(-2)}\,\mathrm{Tr}\,V^{++}\,,$ whereas the second term, $S_{WZ} = \frac{1}{2}\int \mu_A^{(-2)}\mathcal{V}_0\,\widetilde{\mathcal{Z}}^{+a}\mathcal{Z}_a^{+}\,,$ is a WZ action describing coupling of $n$ commuting analytic superfields $\mathcal{Z}_a^{+}$ and the singlet $\mathrm{U}(1)$ part $\mathcal{X}_0 \equiv \mathrm{Tr}\,(\mathcal{X})$. The real analytic superfield $\mathcal{V}_0(\zeta,w)$ is defined by the integral transform for the trace part:

$$\mathcal{X}_0(t,\theta_i,\bar{\theta}^i) = \int dw\Big(1 + m\theta^-\bar{\theta}^+ - m\theta^+\bar{\theta}^- - 2m^2\theta^+\theta^-\bar{\theta}^+\bar{\theta}^-\Big)\mathcal{V}_0\left(t_A,\theta^+,\bar{\theta}^+,w^{\pm}\right)\Big|\,.$$

The $n$ multiplets (4,4,0) are described by the superfields $\mathcal{Z}_a^{+}$ defined by covariantized constraints

$$\nabla^{++}\mathcal{Z}^{+} = \left(\mathcal{D}^{++} + iV^{++}\right)\mathcal{Z}^{+} = 0\,.$$

Using the gauge freedom we can choose the WZ gauge

$$V^{++} = 2i\,\theta^+\bar\theta^+ A(t_A)\,,$$

where now $A(t_A)$ is an $n\times n$ matrix field. In this gauge we have

$$\nabla^{\pm\pm} = \mathcal{D}^{\pm\pm} - 2\,\theta^\pm\bar\theta^\pm\, A, \qquad \nabla^- = \mathcal{D}^- + 2\,\bar\theta^-\, A, \qquad \bar\nabla^- = \bar{\mathcal{D}}^- + 2\,\theta^-\, A\,.$$

After integrating over $\theta$s and harmonics and eliminating auxiliary fields the matrix action in the WZ gauge takes the following form

$$\begin{aligned} S_{matrix} &= S_b + S_f\,, \\ S_b &= \frac{1}{2}\,\mathrm{Tr}\int dt\,\left(\nabla X \nabla X - m^2 X^2\right) - c\int dt\,\mathrm{Tr}A \\ &\quad + \frac{1}{2}\,\mathrm{Tr}\int dt\,\left[ iX_0\left(\nabla\bar Z_k\, Z^k - \bar Z_k \nabla Z^k\right) - \frac{n}{4}\,(\bar Z^{(i} Z^{k)})(\bar Z_i Z_k)\right], \\ S_f &= \frac{1}{2}\,\mathrm{Tr}\int dt\left[ i\left(\bar\Psi_k \nabla\Psi^k - \nabla\bar\Psi_k \Psi^k\right) + 2m\bar\Psi_k\Psi^k\right] - \int dt\,\frac{\Psi_0^{(i}\bar\Psi_0^{k)}(\bar Z_i Z_k)}{X_0}\,, \end{aligned}$$

where

$$X_0 \equiv \mathrm{Tr}(X), \quad \Psi_0^i \equiv \mathrm{Tr}(\Psi^i), \quad \bar\Psi_0^i \equiv \mathrm{Tr}(\bar\Psi^i); \qquad (\bar Z_i Z_k) \equiv \bar Z_i^a Z_{ka}, \quad (\nabla\bar Z_k\, Z^k) \equiv \nabla\bar Z_k^a\, Z_a^k.$$

The covariant derivatives are defined by

$$\nabla X = \dot X + i[A, X]\,, \qquad \nabla\Psi^i = \dot\Psi^i + i[A, \Psi^i]\,, \qquad \nabla\bar\Psi_i = \dot{\bar\Psi}_i + i[A, \bar\Psi_i]\,,$$

$$\nabla Z^k = \dot Z^k + iA\, Z^k\,, \qquad \nabla\bar Z_k = \dot{\bar Z}_k - iA\,\bar Z_k\,.$$

## Bosonic limit in the matrix model

Let us consider the bosonic limit of the matrix action, i.e. the action $S_b$.

Using the residual gauge invariance of the action $S_b$,

$$X' = e^{i\lambda} X e^{-i\lambda}\,, \qquad Z'^k = e^{i\lambda} Z^k\,, \qquad A' = e^{i\lambda} A e^{-i\lambda} - i\, e^{i\lambda}(\partial_t e^{-i\lambda})\,,$$

where $\lambda_a^b(t) \in u(n)$ are ordinary $d{=}1$ gauge parameters, we can impose the gauge

$$X_a^b = 0\,, \qquad a \neq b\,,$$

i.e. $X_a^b = X_a \delta_a^b$ and $X_0 = \sum_{a=1}^{n} X_a$.

As a result of this, and after eliminating $A_a^b$, $a \neq b$, by the equations of motion, the action $S_b$ takes the following form (instead of $Z_a^i$ we introduce the new fields $Z'^i_a = (X_0)^{1/2} Z_a^i$ and omit the primes on these fields),

$$S_b = \frac{1}{2}\int dt \Big\{ \sum_a \Big(\dot X_a \dot X_a - m^2 X_a X_a\Big) - \frac{i}{2}\sum_a \Big(\bar Z_k^a \dot Z_a^k - \dot{\bar Z}{}_k^a Z_a^k\Big) + 2\sum_a A_a^a \Big(Z_k^a Z_a^k - c\Big) + \\ + \sum_{a\neq b} \frac{\mathrm{Tr}(S_a S_b)}{4(X_a - X_b)^2} - \frac{n\,\mathrm{Tr}(\hat S \hat S)}{2(X_0)^2} \Big\},$$

where we used the following notation:

$$(S_a)_k{}^j \equiv \bar Z_k^a Z_a^j, \qquad (\hat S)_k{}^j \equiv \sum_a \Big[(S_a)_k{}^j - \tfrac{1}{2}\delta_k^j (S_a)_l{}^l\Big]$$

and no sum over the repeated index $a$ is assumed.

# Bosonic limit in the matrix model

The terms $\sum_a A^a_a \left(Z^a_k Z^k_a - c\right)$ in the action $S_b$ produce $n$ constraints (for each index $a$)

$$\bar{Z}^a_k Z^k_a - c \approx 0$$

for the fields $Z^k_a$. These constraints generate abelian gauge $[\mathrm{U}(1)]^n$ symmetry, $Z^k_a \to e^{i\varphi_a} Z^k_a$ , with local parameters $\varphi_a(t)$.

Due to these constraints, the fields $Z^k_a$ describe $n$ sets of the target harmonics.

After quantization, these variables become purely internal ($\mathrm{U}(2)$-spin) degrees of freedom.

So, in the Hamiltonian approach, the kinetic WZ term for $Z$ in $S_b$ gives rise to the following Dirac brackets:

$$[Z^k_a, \bar{Z}^b_j]_D = -i\delta^b_a \delta^k_j \,.$$

With respect to these brackets the quantities $S_a$ for each index $a$ form $u(2)$ algebras

$$[(S_a)_i{}^j, (S_b)_k{}^l]_D = i\delta_{ab} \left\{ \delta^l_i (S_a)_k{}^j - \delta^j_k (S_a)_i{}^l \right\}.$$

As a result, after quantization the variables $Z^k_a$ describe $n$ sets of fuzzy spheres.

The action $S_b$ contains a potential in the center-of-mass sector with the coordinate $X_0$. Modulo this extra potential, the bosonic limit of the system constructed is none other than the U(2)-spin Calogero-Moser model which is a massive generalization of the U(2)-spin Calogero model [J. Gibbons, T. Hermsen, 1984; S. Wojciechowski, 1985; A.P. Polychronakos, 1998].

# Conclusion

- We proposed new models of $SU(2|1)$ supersymmetric quantum mechanics as a deformation of the corresponding "flat" $\mathcal{N}=4, d=1$ supersymmetric models.
- The characteristic features of these models is the use of different types of supermultiplets: dynamical, semi-dynamical and pure gauge ones. In considered models, dynamical multiplets are the $(\mathbf{1},\mathbf{4},\mathbf{3})$ ones.
- The prepotential superfield description of the $(\mathbf{1},\mathbf{4},\mathbf{3})$ multiplet has provided an opportunity to build the WZ action for the $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplets and thereby to use the latter for describing semi-dynamical degrees of freedom.
- The $SU(2|1)$ version of the superfield gauging procedure [F. Delduc, E. Ivanov, 2006] involving the appropriate gauge multiplets allowed us to gauge away some of the dynamical and semi-dynamical fields on shell.
- We have studied these new $SU(2|1)$ supersymmetric mechanics models both in the one-particle case and in the multi-particle one. In the latter case the system is described off shell by the matrix theory with $U(n)$ gauging. After elimination of auxiliary and pure gauge fields this matrix theory yields new $\mathcal{N}=4$ superextensions of the $A_{n-1}$ Calogero-Mozer model. The mass (frequency) of the physical states is defined by the deformation parameter of the $SU(2|1)$ supersymmetry.

THANK YOU !